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Conductivity exponents from the analysis of series expansions for random resistor networks

Joan Adler †

Solid State Institute, Technion-Israel Institute of Technology, Haifa 32 000, Israel

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Abstract. There has been considerable controversy in recent years over the value of the conductivity exponent t. This exponent can be deduced from series expansions via the scaling relations, $t = \zeta + (d-2)\nu$, where ζ is deduced from differences between the exponents of the resistive (γ_r), percolative (γ_p) and conductive (γ_c) susceptibilities. We find that allowance for non-analytic confluent corrections to scaling and the use of recent p_c estimates leads to estimates for γ_r, γ_p and γ_c that are somewhat different to those of Fisch and Harris; however, the differences between these exponents do not change significantly. Moreover the change in accepted estimates of ν in the last five years cancels some of this remaining discrepancy and we conclude, (using the relation $\zeta = \gamma_r - \gamma_p$), that $t = 1.31$, $d = 2$; $t = 2.04$, $d = 3$; $t = 2.39$, $d = 4$; $t = 2.72$, $d = 5$; with an error of about ± 0.10 in each case. Our $d = 2$ estimate is in significantly better agreement with those of other methods than that of Fisch and Harris.

In this paper we describe a comprehensive re-examination of extant series expansions for the resistive, percolative and conductive susceptibilities for random resistor networks. These susceptibilities have exponents γ_r, γ_p and γ_c respectively, where

$$\chi_r \sim (p - p_c)^{-\gamma_r} \tag{1}$$

and likewise for χ_p and χ_c. The series were developed by Fisch and Harris (1978, hereafter denoted by FH) who proposed that if one considers L, the average resistance between two connected points and defines the exponent ζ via

$$L \sim (p \sim p_c)^{-\zeta} \tag{2}$$

the exponent t of the conductivity

$$\Sigma \sim (p - p_c)^t \tag{3}$$

could be deduced via the scaling relation

$$t = \zeta + (d-2)\nu, \tag{4}$$

since ζ could be found from the relation,

$$\zeta = \gamma_r - \gamma_p = \gamma_p - \gamma_c. \tag{5}$$

Thus in order to calculate t we require estimates of γ_p, of γ_r or γ_c (or preferably both) and of ν. In order to calculate the γ exponents from the series of FH an estimate of p_c for bond percolation on the hypercubic lattices is necessary and the experience from

† Present address: School of Physics and Astronomy, University of Tel Aviv, Ramat Aviv, Tel Aviv, 69978, Israel.

the two-dimensional (Adler *et al* 1982, Adler *et al* 1983) and three-dimensional (Adler 1984) percolation is that an understanding of the behaviour of non-analytic confluent corrections to scaling is highly desirable. At this point in time better ν estimates for $2 \leq d < 6$, ($d = 2$, Nienhuis 1982, den Nijs 1979, Nienhuis *et al* 1980, Pearson 1980; $d = 3$, Hermann *et al* 1981; $d = 4$ and 5, de Alcantara Bonfim *et al* (1980, 1981), better p_c estimates for $3 \leq d < 6$ ($d = 3$, Wilke 1983; $d = 4$, 5 and 6, Adler *et al* 1984) and better γ_p estimates for $2 \leq d < 6$ ($d = 2$, Nienhuis 1982, den Nijs 1979, Nienhuis *et al* 1980, Pearson 1980; $d = 3$, Herrmann and Stauffer 1981; $d = 4$ and 5 Adler *et al* 1984, de Alcantara Bonfim *et al* 1980, 1981) are available than were extant in 1978. Furthermore, the analysis of FH neglected confluent corrections to scaling since they assumed behaviour of the type of equations (1) and (2). Thus a re-analysis of the FH series that utilises this additional information seems desirable.

We emphasise at the outset that the corrections to scaling considered below are those for the three susceptibility series; we replace equation (1) with

$$\chi_r \sim (p - p_c)^{-\gamma_r}[1 + a_r(p - p_c)^{\Delta_{1r}}] \qquad \text{for } 2 \leq d < 6, \tag{6}$$

and with

$$\chi_r \sim (p - p_c)^{-\gamma_r}[\ln(p - p_c)]^{\theta_r} \qquad \text{for } d = 6. \tag{7}$$

We hypothesise that $\Delta_{1r} = \Delta_{1p} = \Delta_{1c}$, since all 'temperature'-dependent quantities such as the susceptibility, pair correlation length and percolation probability as a function of p usually have the same correction exponent. We do not suggest, however, that if we were to replace equation (3) by $\Sigma \sim (p - p_c)^t[1 + a_t(p - p_c)^{\Delta_{1t}}]$ that Δ_{1t} would be equal to Δ_{1r}; this may or may not be true, but is irrelevant to our analysis. Support for our hypothesis comes from the case of a diode-resistor network where the excellent series for the resistive susceptibility recently developed by Bhatti and Essam (1984) exhibit the same correction behaviour as do the mean cluster size and pair correlation series (Adler *et al* 1981); this will be demonstrated below. With regard to the $6d$ series the situation is somewhat different. Here we have no reasonable basis to assume $\theta_r = \theta_p = \theta_c$, since the exponent of the logarithmic correction varies for 'temperature'-dependent quantities for $6d$ percolation (see p 419 of Adler *et al* 1983). Thus with the aid of the known renormalisation group value of θ_p and γ_p (Essam *et al* 1978) and the new p_c value we shall ask whether $\theta_r = \theta_p = \theta_c$ or not.

We summarise our input data in table 1; we note that the deviation from the three-dimensional γ_p and ν values (1.66 and 0.83) and the two-dimensional γ_p value (2.42) quoted by FH is especially large and *both* γ_p and ν are close to the FH values only at $d = 5$.

The methods of analysis used below have been reviewed in Adler *et al* (1983) ($d < 6$) and Adler *et al* (1984) ($d = 6$). The method developed to analyse behaviour of the type of equation (6) for $d = 2$ and 3 involves transforming the original series in p to one in

$$y = 1 - (1 - p/p_c)^{\Delta}.$$

We then look at different Padé approximants to the function

$$\zeta_\Delta(y) = \Delta(y - 1)(\mathrm{d}/\mathrm{d}y)[\ln \chi(p)] = \gamma - x/(1 + x)$$

where $x = ap_c^{\Delta_1}\Delta_1(y - 1)^{\Delta_1/\Delta}$. The correction term x becomes zero when $p = p_c$ and $\Delta = \Delta_1$. Different Padé approximants to this function are graphed, giving lines of γ as a function of Δ. These should converge near the correct (Δ_1, γ) point for the correct p_c.

Table 1. Recent results for percolation.

Dimension	2	3	4	5	6
p_c	0.5[a]	0.2492 ± 0.0002[b]	0.1603 ± 0.0002[c]	0.1182 ± 0.0002[c]	0.094075 ± 0.0001[c]
ν	$1.333\dot{3}$[d]	0.88 ± 0.01[e]	0.68[f]	0.57[i]	0.5
γ_p	$2.388\dot{8}$[d]	1.74[e]	1.44 ± 0.05[c] 1.44[f]	1.20 ± 0.03[c] 1.18[f]	1
Δ_1	1.25 ± 0.15[g]	1.05 ± 0.15[h]	0.6–1.0[c] 0.88–1.03[i]	0.45–0.9[c] 0.42–0.45[i]	—

[a] Exact (Sykes and Essam 1964).
[b] Monte Carlo (Wilke 1983).
[c] Series (Adler *et al* 1984).
[d] Exact (Nienhuis 1982, den Nijs 1979, Nienhuis *et al* 1980, Pearson 1980).
[e] Monte Carlo (Herrmann *et al* 1981).
[f] Renormalisation Group (de Alcantara Bonfim *et al* 1980, 1981).
[g] Series (Adler *et al* 1983).
[h] Series (Adler 1984).
[i] Renormalisation Group (J Green, private communication)

In this work we use p_c and Δ_1 as input values, since p_c for $d = 2$ and 3 is available to a higher precision than it is possible to generate from series and p_c for $d = 4$ and 5 and Δ_1 for all d has been estimated recently from series that are longer than those of FH. We note that for some of the series studied below we do not find clear convergence regions, however we use the Δ_1 estimates to obtain γ_r and γ_c valves for the input p_c values.

Our overall results are summarised in table 2. In the first row we present the results of an analysis of the percolative suceptibility series. Comparison of this row with the

Table 2. Results from analysis of FH series and comparison of t values.

Dimension	2	3	4	5
Results of re-analysis				
γ_p[a]	2.37 ± 0.10	1.80 ± 0.04	1.45 ± 0.08	1.19 ± 0.03
γ_r	3.70 ± 0.20	2.90 ± 0.10	2.47 ± 0.10	2.20 ± 0.05
γ_c	0.98 ± 0.04	0.66 ± 0.04	0.41 ± 0.08	0.3 ± 0.01
ζ[b]	1.36 ± 0.12	1.12 ± 0.07	1.03 ± 0.09	0.95 ± 0.08
ζ[c]	1.31	1.16	1.03	1.01
t[d]	1.36 ± 0.12	2.00 ± 0.08	2.39	2.66
t[e]	1.31	2.04	2.39	2.72
Results for comparison				
t	1.264[f] 1.3[g]	1.98[f] 1.94 ± 0.1[h] 2.2[i]		
ζ[j]	1.43	1.12	1.05	1.02
t[j]	1.43	1.95	2.37	2.73

[a] Results from the FH series, presented for comparison.
[b] $\zeta = (\gamma_r - \gamma_c)/2$.
[c] $\zeta = (\gamma_r - \gamma_p)$, γ_p from table 1, error as for [b].
[d] $t = \zeta + (d-2)\nu$, ζ as in [b], ν table 1, error $\geqslant$ error for [b].
[e] $t = \zeta + (d-2)\nu$, ζ as in [c], ν table 1, error $\geqslant$ error for [b].
[f] Alexander-Orbach conjecture; exact β for 2D, β of Adler (1984) for 3D, ν from table 1.
[g] Zabolitsky (1984) Monte Carlo. [h] Derrida *et al* (1983) transfer matrix.
[i] Mitescu and Greene (1983). [j] Fisch and Harris (1978).

γ_p row in table 1 suggests that for 4*d* and 5*d* the agreement with Adler *et al* (1984) is excellent and even for 2*d* and 3*d* agreement with exact and Monte Carlo values respectively, is reasonable. These γ_p results are quoted for comparison only; we use the γ_p values of table 1 below. In the second and third row we present our estimates for γ_r and γ_c. These estimates include all γ values corresponding to the p_c and Δ_1 estimates of table 1. The $(\Delta_1 \gamma)$ plane of the central p_c estimates are illustrated for $d = 2$, 3, 4 and 5 in figures 1, 2, 3 and 4 respectively. We indicate the Δ_1 estimate of table 1 by a bar; should these estimates be revised in the future, new γ_r and γ_c values could be read off the graphs. We obtain estimates of ζ and t both from $\zeta = (\gamma_r - \gamma_c)/2$ and from $\zeta = (\gamma_r - \gamma_p)$. It is not clear which expression is the more reliable; since

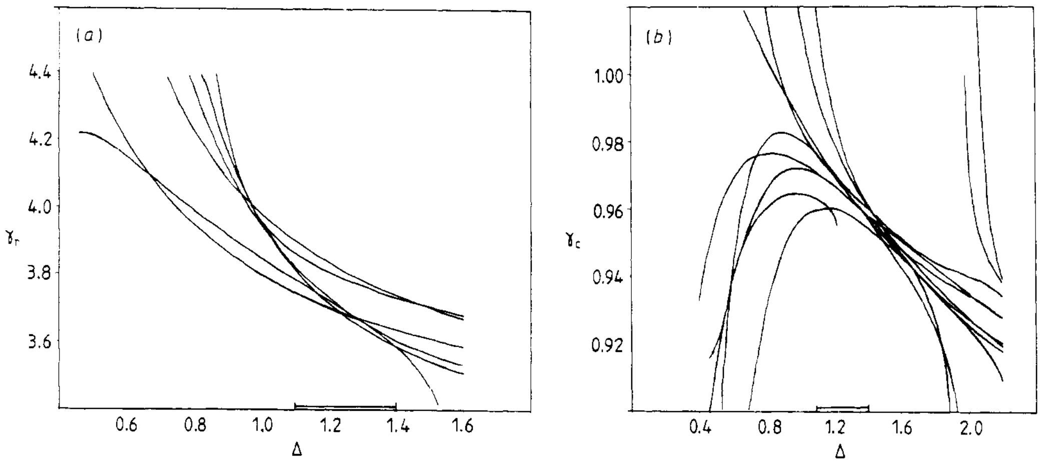


Figure 1. Graphs of Padé approximants to (*a*) γ_r, (*b*) γ_c as a functions of Δ for 2D bond percolation on the square lattice at $p = 0.5$.

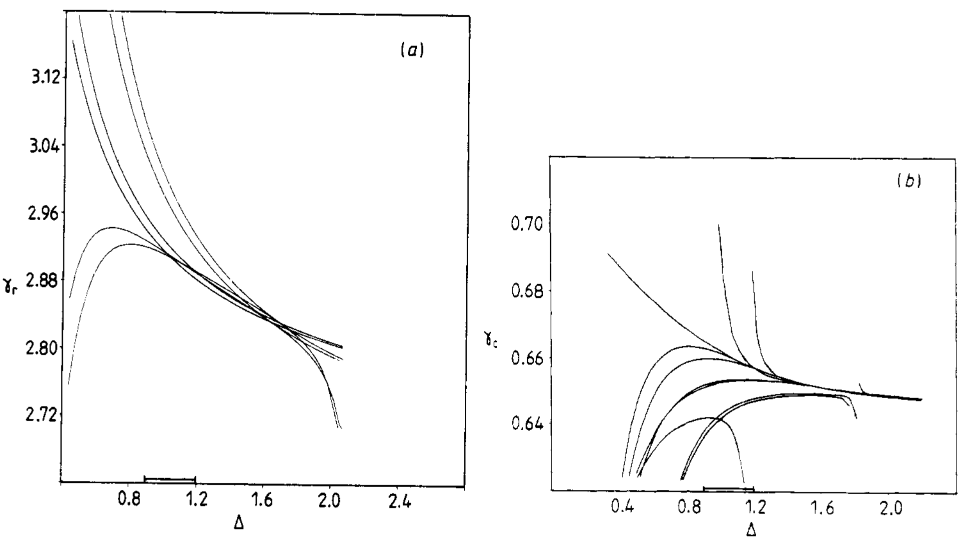
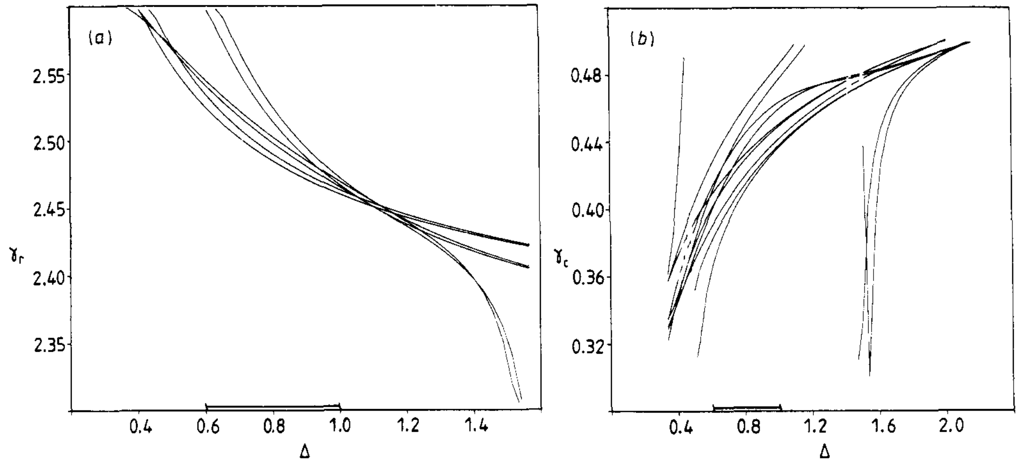


Figure 2. Graphs of Padé approximants to (*a*) γ_r, (*b*) γ_c as functions of Δ for 3D bond percolation on the SC lattice at $p = 0.2492$.



13070 **Figure 3.** Graphs of Padé approximants to (*a*) γ_r, (*b*) γ_c as functions of Δ for 4D bond
13071 percolation on the hypercubic lattice at $p = 0.1603$.

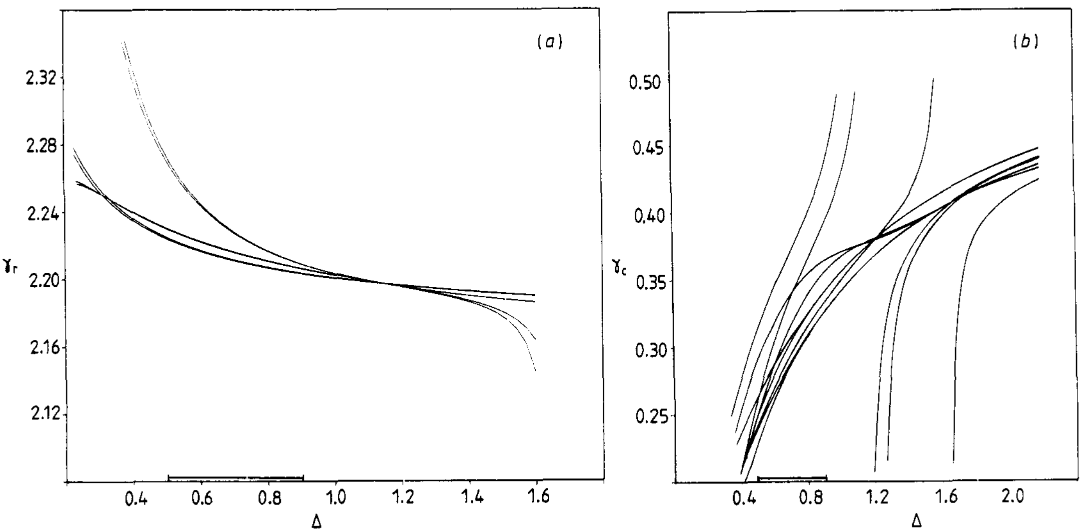


Figure 4. Graphs of Padé approximants to (*a*) γ_r, (*b*) γ_c as functions of Δ for 5D bond percolation on the hypercubic lattice at $p = 0.1182$.

$\gamma_c \to 0$ as $d \to 6$ we expect that the latter will be more reliable near $d = 6$, as Padé-type analyses are less reliable for very small exponents.

It could be assumed that the latter is also more reliable near $d = 2$, since γ_p is known to higher accuracy than γ_c. Here, however, a difference between two values calculated from similar data could be freer of possible systematic errors. Thus we include both estimates; they are close everywhere except at $d = 5$ (where we may claim that our γ_c is unreasonably large owing to problems with Padé, (FH obtained a lower γ_c value, and they used ratio as well) and at $d = 2$, where we could again assume that the γ_c value is the inconsistent one. Inspection of figure 1(*b*) does not, however, give any reason to justify a γ_c value > 1.02, thus this assumption does not appear to be justifiable. We note that the Padé approximants presented in the (*a*) figures are the [3, 4], [4, 3], [2, 4], [4, 2], [2, 3], [3, 2], [1, 3] and [2, 2] approximants; in the (*b*) figures we used the [2, 5], [3, 4], [4, 3], [5, 2], [2, 4], [3, 3], [4, 2], [2, 3] and [3, 2] approximants.

The results discussed above depend on the hypothesis that $\Delta_{1r} = \Delta_{1p} = \Delta_{1c}$. As indicated above, support for this hypothesis comes from the case of directed percolation; we present the (γ_p, Δ) plane for the mean cluster size series of De'Bell and Essam (1983) in figure 5(*b*) and the (γ_r, Δ) plane for the resistive susceptibility of Bhatti and Essam (1984) in figure 5(*a*). For both series $\Delta_2 = 1.0 \pm 0.1$ (consistent with Adler *et al* 1981) and we can see that the nature of the confluent corrections to scaling is quite similar, both appearing to be analytic.

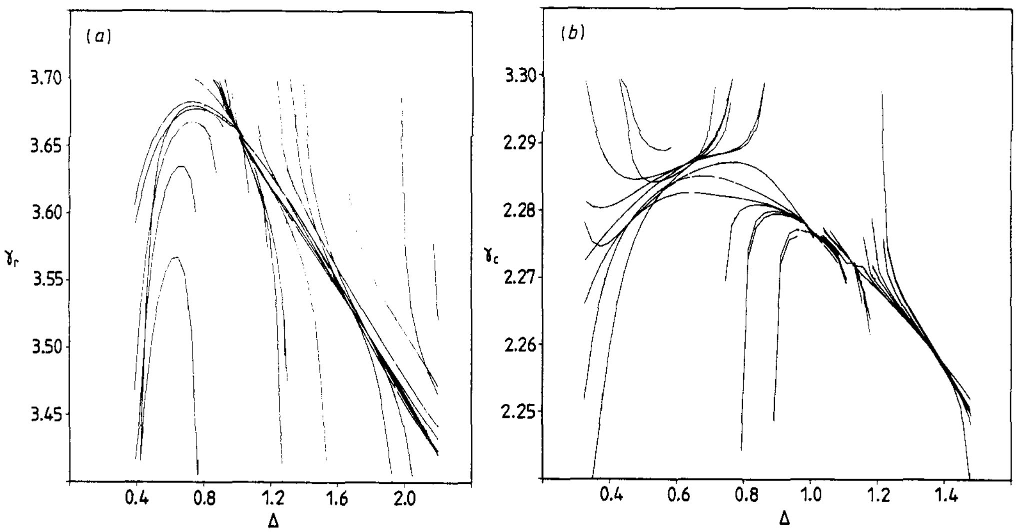


Figure 5. Graphs of Padé approximants to (*a*) γ_r, (*b*) γ_c as functions of Δ for 2D directed bond percolation on the square lattice at $p = 0.644701$ (De'Bell, private communication).

For behaviour of the equation (7) type we use the method of Adler and Privman (1981). This method was developed to prove the absence of logarithmic corrections in $d = 2$ percolation, but is equally suitable for demonstrating their presence. We write $\theta = z\gamma$ and derive the series for

$$g(p) = (1/-\gamma)(p-p_c)\ln[p_c-p)(\chi^1(p)/\chi(p) - \gamma/(p_c-p)].$$

We can show that

$$\lim_{p \to p_c} g(p) = z$$

and form Padé approximants to $g(p)$ in order to evaluate θ. We graph γ as a function of θ for different p_c values, and note that the θ value is extremely sensitive to p_c. Since for $d = 6$ we know that $\gamma_r = 2$ and $\gamma_c = 0$, the main interest here is to determine whether $\theta_r = \theta_p$. We present graphs of the Padé approximants to $g(p)$ in figures 6(*a*) and 6(*b*) for χ_r and χ_p respectively for $p_c = 0.094025$. The RG exponents $\gamma_p = 1$, $z_p = \frac{2}{7}$ are indicated in figure 6(*b*) by an asterisk, and a diamond illustrates the point $\gamma_r = 1$, $\theta_r = \frac{2}{7}$ in figure 6(*a*). From the strong similarity between the two graphs we may conjecture that the correct result is that $\theta_r = \theta_p$.

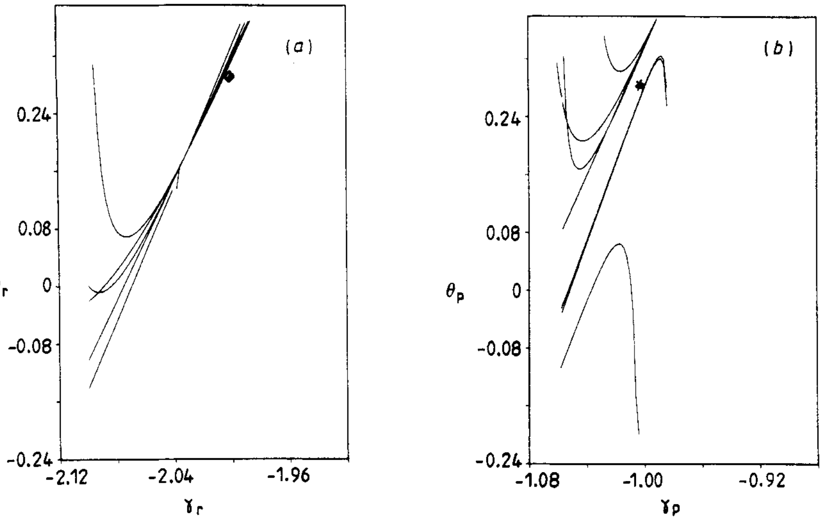


Figure 6. Graphs of Padé approximants to (a) θ_r, (b) θ_p as functions of γ_r and γ_p respectively for 6D bond percolation on the hypercubic lattice at $p = 0.094025$.

Finally we shall compare our results with existing estimates, which are summarised in the latter part of table 2. We see that for $d = 4$ and 5 our final t values are fairly close to FH, differences in γ_r and ν values having cancelled each other out in the case of $d = 4$. For d = 3, where we find $\gamma_r = 2.9$, $\gamma_p = 1.8$ and use $\nu = 0.88$, FH found $\gamma_r = 2.78$, $\gamma_p = 1.66$ and used $\nu = 0.83$, however, the final results differ only by 0.06 and their central value is closer to the other estimates, although our result is consistent with the other values listed in table 2. For $d = 3$, 4 and 5 differences between our results and those of FH are mainly due to differences in p_c values. At $d = 2$ our γ values differ more markedly from those of FH and looking at figure 1(a) we can see that the Padé aproximants γ_c as a function of Δ slope quite strongly. The FH value of γ_r (=3.8) corresponds to $\Delta = 1$, (the result to be expected if non-analytic confluent corrections to scaling are neglected and the value near $\Delta_1 \sim 1.25$ is clearly below 3.8. Our γ_c value and that of FH ($\gamma_c = 0.99$) are similar; here the Padé approximants can be observed (figure 1(b)) to be relatively flat. Our value of the conductivity exponent ($t = 1.31$), calculated using $\zeta = \gamma_r - \gamma_p$ is in excellent agreement with Zabolitsky (1984) and even our value deduced from $\zeta = (\gamma_r - \gamma_c)/2$ ($t = 1.36$) is in better agreement with Zabolitsky's value than is FH. If we compare the ζ values calculated from these two relations using the FH values ($\gamma_p = 2.42$) we obtain $t = 1.38$ and $t = 1.41$ respectively, and only by using $\zeta = \gamma_p - \gamma_c$ does one have $t = 1.43$ which is the value they quote. We may thus observe that there are two reasons why the t value of FH at $d = 2$ is so much higher than all other estimates from the literature. One is the lack of consideration of confluent corrections to scaling (which explains why the γ_r and γ_p values of FH are above our estimate and the exact result, respectively) and the other is the apparent choice of $\zeta = \gamma_p - \gamma_c$, rather than either $\zeta = \gamma_r - \gamma_p$ or $\zeta = (\gamma_r - \gamma_c)/2$.

In conclusion, we have re-analysed the FH series to find new estimates of γ_r, γ_c and t. Our new estimates agree with FH except at $d = 3$, where the difference is small and at $d = 2$ where the difference is larger and our result is much closer to estimates from other calculations.

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After completing this calculation we received preprints from M Sahimi and from A Aharony and D Stauffer. Sahimi found $\gamma_r(d=2)=3.72$ which is in excellent agreement with our value and his conjectures for $t\,(d>2)$ are also very close to our estimates. Aharony and Stauffer conjecture that $t=\frac{4}{3}\,(d=2)$ and this is consistent with our results. This work was supported in part by a grant from the Israel-U.S. Binational Science Foundation.

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